

Intrinsic Ultracontractivity on Riemannian Manifolds with Infinite Volume Measures

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February 2, 2008

Abstract

By establishing the intrinsic super-Poincaré inequality, some explicit conditions are presented for diffusion semigroups on a non-compact complete Riemannian manifold to be intrinsically ultracontractive. These conditions, as well as the resulting uniform upper bounds on the intrinsic heat kernels, are sharp for some concrete examples.

AMS subject Classification: 58G32, 60J60

Keywords: Intrinsic ultracontractivity, intrinsic super-Poincaré inequality, Riemannian manifold, diffusion semigroup.

1 Introduction

Let (E, \mathcal{F}, μ) be a σ -finite measure space, and $(L, \mathcal{D}(L))$ a negative self-adjoint operator generating a (sub-)Markov semigroup $P_t := e^{tL}$ on $L^2(\mu)$. According to [8], the

*Supported in part by Creative Research Group Fund of the National Foundation of China (No. 10121101), RFDP(20040027009) and the 973-project in China.

semigroup P_t is called ultracontractive if $\|P_t\|_{L^1(\mu) \rightarrow L^\infty(\mu)} < \infty$ for any $t > 0$. Due to Gross [10], the ultracontractivity of P_t can be derived from log-Sobolev inequalities and for μ finite, implies that P_t is compact and L has empty essential spectrum (see [20] for more general results on functional inequalities and the essential spectrum). On the other hand, however, when μ is infinite, there is no direct relationship between the ultracontractivity and the spectrum of L . For instance, the heat semigroup on \mathbb{R}^d is ultracontractive with respect to the Lebesgue measure, but the spectrum of $-\Delta$ is continuous. This means that the ultracontractivity is no longer “intrinsic” for the spectrum property when μ is infinite. For this reason and for other applications in the study of Markov semigroups, the intrinsic ultracontractivity was introduced (cf. [8]).

Assume that $\lambda_0 := \inf \sigma(-L)$ is a simple eigenvalue of $-L$ with $\varphi_0 > 0$ the unique unit eigenfunction, where and in what follows, we use $\sigma(\cdot)$ and $\sigma_{ess}(\cdot)$ to denote the spectrum and the essential spectrum of an operator respectively. In general, for the case that λ_0 is an eigenvalue, we may always take a nonnegative eigenfunction φ_0 (called the ground state in the literature). Indeed, let φ be an eigenfunction with respect to λ_0 with $\mu(\varphi > 0) > 0$ (otherwise, use $-\varphi$ in stead of φ), we have

$$P_t \varphi^+ \geq (P_t \varphi)^+ = e^{-\lambda_0 t} \varphi^+$$

for all $t \geq 0$. But due to the definition of λ_0 , one has

$$\|P_t \varphi^+\|_{L^2(\mu)} \leq e^{-\lambda_0 t} \|\varphi^+\|_{L^2(\mu)},$$

then $P_t \varphi^+ = e^{-\lambda_0 t} \varphi^+$ for any $t \geq 0$, so that φ^+ is an eigenfunction with respect to λ_0 too. If in addition that P_t is irreducible in the sense that $\mu(1_A P_t 1_B) > 0$ for any $A, B \in \mathcal{F}$ with $\mu(A)\mu(B) > 0$, then the nonnegative eigenfunction has to be strictly positive and λ_0 is a simple eigenvalue. In this case, $\mu_{\varphi_0} := \varphi_0^2 \mu$ is a probability measure and

$$P_t^{\varphi_0} f := \frac{e^{\lambda_0 t}}{\varphi_0} P_t(f \varphi_0), \quad t \geq 0, f \in L^2(\mu_{\varphi_0})$$

gives rise to a symmetric C_0 Markov semigroup on $L^2(\mu_{\varphi_0})$. We call P_t intrinsically ultracontractive if

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})} < \infty, \quad t > 0.$$

Moreover, P_t is called intrinsically hypercontractive if there exists $t > 0$ such that

$$\|P_t^{\varphi_0}\|_{L^2(\mu_{\varphi_0}) \rightarrow L^4(\mu_{\varphi_0})} = 1.$$

The intrinsic ultracontractivity has been well studied in the framework of Dirichlet heat semigroups on (in particular, bounded) domains in \mathbb{R}^d . For instance, the Dirichlet

heat semigroup on a bounded Hölder domain of order 0 is intrinsically ultracontractive (see [5, 3]). See the recent work [16] and references within for sharp estimates on $\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})}$, and [13] and references within for the study of the intrinsic ultracontractivity of Lévy (in particular, stable) processes on domains.

In this paper, we aim to study the intrinsic ultracontractivity for diffusion semigroups on Riemannian manifolds with infinite invariant measures. Let M be a complete, connected, non-compact Riemannian manifold of dimension d . Let $L = \Delta + \nabla V$ for some $V \in C^2(M)$. Then L generates a unique (Dirichlet) diffusion semigroup P_t on M which is symmetric in $L^2(\mu)$, where $\mu := e^{V(x)} dx$ for dx the Riemannian volume measure. Assume that $\lambda_0 := \inf \sigma(-L)$ is an eigenvalue of $-L$. Since M is connected, λ_0 has a unique unit eigenfunction $\varphi_0 > 0$.

To clarify the meaning in geometry analysis of the intrinsic ultracontractivity, let us recall that

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})} = \sup_{x, y \in M} \frac{e^{\lambda_0 t} h(x, y, t)}{\varphi_0(x) \varphi_0(y)}, \quad t > 0,$$

where $h(x, y, t)$ is the heat kernel of P_t with respect to the weighted volume measure μ .

In order to study the intrinsic ultracontractivity of P_t , we make use of the following intrinsic super-Poincaré inequality introduced in [21] (see also [16]):

$$(1.1) \quad \mu(f^2) \leq r\mu(|\nabla f|^2) + \beta(r)\mu(\varphi_0|f|)^2, \quad r > 0, f \in C_0^1(M),$$

where $\beta : (0, \infty) \rightarrow (0, \infty)$ is a decreasing function and

$$\mu(f) := \int_M f d\mu, \quad f \in L^1(\mu).$$

The intrinsic ultracontractivity of P_t implies (1.1) for some β (see [21, Theorem 3.1]), and (1.1) holds for some β if and only if $\sigma_{ess}(L) = \emptyset$ (see [21, Theorem 2.2]). On the other hand, if

$$(1.2) \quad \Psi(t) := \int_t^\infty \frac{\beta^{-1}(s)}{s} ds < \infty, \quad t > \inf_{r>0} \beta(r),$$

where $\beta^{-1}(s) := \inf\{r > 0 : \beta(r) \leq s\}$ for a positive decreasing function β , then (1.1) implies the intrinsic ultracontractivity of P_t with (see [20, Theorem 3.3])

$$(1.3) \quad \|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})} \leq \max\{\varepsilon^{-1} \inf \beta, \Psi^{-1}((1-\varepsilon)t)\}^2 < \infty, \quad \varepsilon \in (0, 1), t > 0.$$

We refer to [8] for the study of intrinsic ultracontractivity using the log-Sobolev inequality with parameters.

In section 2, (1.1) with explicit β is established in terms of curvature lower bounds of L , the first Dirichlet eigenvalue of L outside large balls, and the super Poincaré inequality on M (see Theorem 2.1 below). As applications, we obtain the following two concrete results (see Section 3 for complete proofs).

To state these results, we introduce some curvature conditions. Let Sec and Ric denote the sectional curvature and the Ricci curvature on M respectively. Let ρ be the Riemannian distance on M , and simply write $\rho_o := \rho(o, \cdot)$ for a fixed reference point $o \in M$. Let k and K be two positive increasing functions on $[0, \infty)$ such that

$$(1.4) \quad \text{Sec} \leq -k \circ \rho_o, \quad \text{Ric} \geq -K \circ \rho_o, \quad \rho_o \gg 1$$

holds on M . Here $\text{Sec} \leq -k \circ \rho_o$ means that for any $x \in M$ and any unit vectors $X, Y \in T_x$ with $\langle X, Y \rangle = 0$, one has $\text{Sec}(X, Y) \leq -k(\rho_o(x))$; while $\text{Ric} \geq -K \circ \rho_o$ means that $\text{Ric}(X, X) \geq -K(\rho_o(x))|X|^2$ for any $x \in M$ and $X \in T_x$. Finally, for a positive increasing function h on $(0, \infty)$, we let

$$h^{-1}(r) := \inf\{s > 0 : h(s) \geq r\}, \quad r \geq 0.$$

Theorem 1.1. *Let M be a Cartan-Hadamard manifold with $d \geq 2$ and let $L = \Delta$. Assume that (1.4) holds for some positive increasing functions k and K with $k(\infty) = \infty$. We have:*

(1) (1.1) holds with

$$\beta(r) := \theta r^{-d/2} \exp \left[\theta k^{-1}(\theta/r) \sqrt{K(4 + 2k^{-1}(\theta/r))} \right], \quad r > 0$$

for some constant $\theta > 0$.

(2) If

$$(1.5) \quad k^{-1}(R) \sqrt{K(4 + 2k^{-1}(R))} \leq cR^\varepsilon, \quad R \gg 1$$

holds for some constants $c > 0$ and $\varepsilon \in (0, 1)$, then P_t is intrinsically ultracontractive with

$$(1.6) \quad \|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})} \leq \exp \left[C(1 + t^{-\varepsilon/(1-\varepsilon)}) \right], \quad t > 0$$

for some constant $C > 0$, or equivalently

$$(1.7) \quad h(x, y, t) \leq e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y) \exp \left[C(1 + t^{-\varepsilon/(1-\varepsilon)}) \right], \quad x, y \in M, t > 0.$$

(3) If (1.5) holds for some $c > 0$ and $\varepsilon = 1$, then P_t is intrinsically hypercontractive.

Remark 1.1. (a) If $\text{Ric} \geq -K$ for some constant $K \geq 0$, then $\sigma_{\text{ess}}(\Delta) \neq \emptyset$. Since M is non-compact and complete, this follows from a comparison theorem by Cheng [4] for the first Dirichlet eigenvalue and the Donnelly-Li decomposition principle [9] for the essential spectrum:

$$\inf \sigma_{\text{ess}}(-\Delta) \leq \sup_{x \in M} \lambda_0(B(x, 1)) \leq \lambda_0(K),$$

where $\lambda_0(B(x, 1))$ is the first Dirichlet eigenvalue of $-\Delta$ on D and $\lambda_0(K)$ is the one on the unit geodesic ball in the d -dimensional parabolic space with Ricci curvature equal to K . Thus, the assumption $K(\infty) = \infty$ in Theorem 1.1 is necessary for (1.1) to hold. Correspondingly, the assumption that $k(\infty) = \infty$ is also reasonable.

(b) The upper bound given in (1.6), which is sharp due to Example 1.1 below, is quite different from the known one on bounded domains. Indeed, for P_t the Dirichlet heat semigroup on a bounded $C^{1,\alpha}$ ($\alpha > 0$) domain in \mathbb{R}^d , the short time behavior of the intrinsic heat kernel is algebraic rather than exponential (see [16]):

$$\sup_{x,y} \frac{h(x,y,t)e^{\lambda_0 t}}{\varphi_0(x)\varphi_0(y)} = \mathcal{O}(t^{-(d+2)/2}).$$

Next, we consider the case with drift. To this end, we adopt the following Bakry-Emery curvature $\text{Ric}_{m,L}$ instead of Ric . Assume that for some $m > 0$ and positive increasing function K one has, instead of the second condition in (1.4),

$$(1.8) \quad \text{Ric}_{L,m} := \text{Ric} - \text{Hess}_V - \frac{\nabla V \otimes \nabla V}{m} \geq -K \circ \rho_o.$$

Moreover, let γ be a positive increasing function on $[0, \infty)$ such that

$$(1.9) \quad L\rho_o \geq \sqrt{\gamma \circ \rho_o}, \quad \rho_o \gg 1.$$

Theorem 1.2. *Let o be a pole in M such that (1.8) and (1.9) hold for some increasing positive functions K and γ with $\gamma(\infty) = \infty$. Then $\sigma_{\text{ess}}(L) = \emptyset$. Moreover, assuming*

$$(1.10) \quad \lim_{\rho_o(x) \rightarrow \infty} \frac{\sqrt{K(2 + 2\rho_o(x))}}{\log^+ \mu(B(x, 1))} = 0,$$

where $B(x, 1)$ is the unit geodesic ball at x , we have:

(1) (1.1) holds with

$$\beta(r) = \theta r^{-(m+d+1)/2} \exp \left[\theta \gamma^{-1}(32/r) \sqrt{K(2 + 2\gamma^{-1}(32/r))} \right], \quad r > 0$$

for some constant $\theta > 0$.

(2) If there exist $c > 0$ and $\varepsilon \in (0, 1)$ such that

$$(1.11) \quad \gamma^{-1}(R) \sqrt{K(2 + 2\gamma^{-1}(R))} \leq cR^\varepsilon, \quad R \gg 1,$$

then P_t is intrinsically ultracontractive with (1.6) and (1.7) holding for some constant $C > 0$.

(3) If (1.11) holds for some $c > 0$ and $\varepsilon = 1$, then P_t is intrinsically hypercontractive.

To conclude this section, we present below two typical examples to show that conditions in Theorems 1.1 and 1.2 are sharp. To make the introduction brief, we leave their proofs to Section 4.

Example 1.1. Let M be a Cartan-Hadamard manifold with

$$-c_1 \rho_o^\delta \leq \text{Sec} \leq -c_2 \rho_o^\delta, \quad \rho_o \gg 1$$

for some constants $c_1, c_2, \delta > 0$. Then $\sigma_{ess}(\Delta) = \emptyset$ and for $L = \Delta$, (1.1) holds with

$$\beta(r) = \exp[c(1 + r^{-(2+\delta)/[2\delta]})]$$

for some constant $c > 0$. Consequently:

(1) P_t is intrinsically ultracontractive if and only if $\delta > 2$, and when $\delta > 2$ one has

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})} \leq \theta_1 \exp [\theta_2 t^{-(\delta+2)/(\delta-2)}], \quad t > 0$$

for some constants $\theta_1, \theta_2 > 0$, which is sharp in the sense that the constant θ_2 can not be replaced by any positive function $\theta_2(t)$ with $\theta_2(t) \downarrow 0$ as $t \downarrow 0$.

(2) P_t is intrinsically hypercontractive if and only if $\delta \geq 2$.

Example 1.2. Let M be a Cartan-Hadamard manifold with

$$\text{Ric} \geq -c(\rho_o^{2(\delta-1)} + 1)$$

for some constants $c > 0$ and $\delta > 1$. Let $V = \theta\rho_o^\delta$ for some constant $\theta > 0$ and $\rho_o \gg 1$. Then $\sigma_{ess}(L) = \emptyset$ and (1.1) holds with

$$\beta(r) = \exp[c(1 + r^{-\delta/[2(\delta-1)]})]$$

for some constant $c > 0$. Consequently:

- (1) P_t is intrinsically ultracontractive if and only if $\delta > 2$, and when $\delta > 2$ one has

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})} \leq \theta_1 \exp[\theta_2 t^{-\delta/(\delta-2)}], \quad t > 0$$

for some constants $\theta_1, \theta_2 > 0$, which is sharp in the sense that the constant θ_2 can not be replaced by any positive function $\theta_2(t)$ with $\theta_2(t) \downarrow 0$ as $t \downarrow 0$.

- (2) P_t is intrinsically hypercontractive if and only if $\delta \geq 2$.

2 The intrinsic super-Poincaré inequality

As explained in the last section that due to [21, Theorem 2.2], (1.1) holds for some β if and only if $\sigma_{ess}(L) = \emptyset$. According to the Donnelly-Li decomposition principle (see [9]), they are also equivalent to

$$\lambda_0(R) := \inf\{\mu(|\nabla f|^2) : \mu(f^2) = 1, f \in C_0^1(M), f|_{B(o,R)} = 0\} \uparrow \infty$$

as $R \uparrow \infty$. The purpose of this section is to estimate β in (1.1) by using $\lambda_0(R)$ and the curvature condition. To this end, we will make use of the following super-Poincaré inequality:

$$(2.1) \quad \mu(f^2) \leq r\mu(|\nabla f|^2) + \beta_0(r)\mu(|f|)^2, \quad r > 0, f \in C_0^1(M)$$

for some decreasing function $\beta_0 : (0, \infty) \rightarrow (0, \infty)$. In particular, by [20, Corollary 1.1 (2)], (2.1) with $\beta_0(r) = c(1 + r^{-p/2})$ for some constant $c > 0$ and $p > 2$ is equivalent to the classical Sobolev inequality

$$\mu(|f|^{2p/(p-2)})^{(p-2)/p} \leq C(\mu(|\nabla f|^2) + \mu(f^2)), \quad f \in C_0^1(M)$$

for some constant $C > 0$. The latter inequality holds for a large class of non-compact manifolds. For instance, according to [6], it holds true for $V = 0$ provided either the injectivity radius of M is infinite, or the injectivity radius is positive and the Ricci curvature is bounded below.

Theorem 2.1. Assume (2.1). Let K be positive increasing function on $[0, \infty)$ such that (1.8) holds. If $\lambda_0(R) \uparrow \infty$ as $R \uparrow \infty$, then (1.1) holds with

$$\beta(r) = C\beta_0(r/8) \exp \left[C\lambda_0^{-1}(8/r)K(2 + 2\lambda_0^{-1}(8/r)) \right], \quad r > 0.$$

To prove this result, we first estimate the ground state φ_0 from below. The following lemma is proved by using the Li-Yau ([14]) type parabolic Harnack inequality derived by X.-D. Li [15].

Lemma 2.2. If (1.8) holds then for the positive ground state φ_0 , there exists a constant $C > 0$ such that

$$\varphi_0 \geq \frac{1}{C} \exp \left[-C\rho_o \sqrt{K(2\rho_o)} \right].$$

Proof. Since φ_0 is bounded below by a positive constant on a compact set, it suffices to prove for $\rho_o \geq 1$. Let $x \in M$ with $\rho_o(x) \geq 1$. Applying [15, Theorem 5.2] to $\alpha = 2$ and $R = \rho_o(x)$, we obtain

$$\begin{aligned} e^{-\lambda_0}\varphi_0(o) &= P_1\varphi_0(o) \leq (P_{1+s}\varphi_0(x))(1+s)^{m+d} \exp \left[c_1^2 s K(2\rho_o(x)) + \frac{\rho_o(x)^2}{2s} \right] \\ &= \varphi_0(x) e^{-\lambda_0(1+s)} (1+s)^{m+d} \exp \left[c_1^2 s K(2\rho_o(x)) + \frac{\rho_o(x)^2}{2s} \right], \quad s > 0 \end{aligned}$$

for some constant $c_1 > 0$. Then the proof is completed by taking $s = \rho_o(x)/\sqrt{K(2\rho_o(x))}$. \square

Proof of Theorem 2.1. Since one may always take decreasing β , it suffices to prove for $r \leq 1$. Let $f \in C_0^1(M)$ be fixed. Let $h_R = (\rho_o - R)^+ \wedge 1$, $R > 0$. Then h_R is Lipschitz continuous so that (2.1) applies to $f(1 - h_R)$ instead of f :

$$(2.2) \quad \mu(f^2(1 - h_R)^2) \leq 2s\mu(|\nabla f|^2) + 2s\mu(f^2) + \beta_0(s)\mu(|f|1_{B(o, R+1)})^2, \quad s > 0.$$

Next, since $h_R f = 0$ on $B(o, R)$, we have

$$\mu(f^2 h_R^2) \leq \frac{\mu(|\nabla(fh_R)|^2)}{\lambda_0(R)} \leq \frac{2}{\lambda_0(R)}\mu(|\nabla f|^2) + \frac{2}{\lambda_0(R)}\mu(f^2).$$

Combining this with (2.2) we obtain

$$\begin{aligned} \mu(f^2) &\leq 2\mu(f^2 h_R^2) + 2\mu(f^2(1 - h_R)^2) \\ &\leq \left(4s + \frac{4}{\lambda_0(R)} \right) (\mu(|\nabla f|^2) + \mu(f^2)) + \frac{2\beta_0(s)}{\inf_{B(o, R+1)} \varphi_0^2} \mu(|f|\varphi_0)^2. \end{aligned}$$

Thus, if $4s + \frac{4}{\lambda_0(R)} \leq \frac{1}{2}$ then

$$\mu(f^2) \leq \left(8s + \frac{8}{\lambda_0(R)}\right) \mu(|\nabla f|^2) + \frac{4\beta_0(s)}{\inf_{B(o, R+1)} \varphi_0^2} \mu(|f|\varphi_0)^2.$$

Hence, (1.1) holds for

$$\beta(r) := \inf \left\{ \frac{4\beta_0(s)}{\inf_{B(o, R+1)} \varphi_0^2} : 8s + \frac{8}{\lambda_0(R)} \leq r \right\}, \quad r \leq 1.$$

Combining this with Lemma 2.2, there exists a constant $c > 0$ such that (1.1) holds for

$$\beta(r) := \inf \left\{ c\beta_0(s) \exp [c(R+1)\sqrt{K(2+2R)}] : 8s + \frac{8}{\lambda_0(R)} \leq r \right\}, \quad r \leq 1.$$

This completes the proof by taken $s = r/8$ and $R = \lambda_0^{-1}(8/r)$. \square

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. a) Since $V = 0$, (1.4) implies (1.8). Moreover, since M is a Cartan-Hadamard manifold, its injectivity radius is infinite. Hence, by [6], one has $\|P_t\|_{L^1(\mu) \rightarrow L^\infty(\mu)} \leq ct^{-d/2}$ for some $c > 0$ and all $t > 0$. By [20, Theorem 4.5 (b)], this implies (2.1) with $\beta_0(r) = c(1 + r^{-d/2})$ for some constant $c > 0$.

b) Since M is a Cartan-Hadamard manifold, $B(o, R)^c$ is concave. Let $R_0 > 0$ be such that (1.4) holds for $\rho_o \geq R_0$. Then for any $R \geq R_0$, we have $\text{Sec} \leq -k(R)$ on $B(o, R)^c$. To make use of the Laplacian comparison theorem, we note that the distance to the boundary of $B(o, R)^c$ is $\rho_o - R$ for $\rho_o \geq R$, and that the function

$$h(s) := \cosh(\sqrt{k(R)}s), \quad s \geq 0$$

solves the equation

$$h''(s) - k(R)h(s) = 0, \quad h(0) = 1, \quad h'(0) = 0.$$

Then (see [12, Theorem 0.3])

$$(3.1) \quad \Delta\rho_o \geq \frac{(d-1)h'(\rho_o - R)}{h(\rho_o - R)} \geq c_0\sqrt{k(R)}, \quad \rho_o \geq R+1$$

holds for some constant $c_0 > 0$ which is independent of R . By the Green formula, for any smooth domain $D \subset B(o, R+1)^c$, it follows from (3.1) that

$$c_0 \sqrt{k(R)} \mu(D) \leq \int_D \Delta \rho_o d\mu \leq \int_{\partial D} |N \rho_o| d\mu_{\partial} \leq \mu_{\partial}(\partial D),$$

where N is the unit normal vector field on ∂D and μ_{∂} is the measure on ∂D induced by μ . Thus, by Cheeger's inequality (see [2]), we have

$$\lambda_0(R+1) \geq \frac{c_0^2 k(R)}{4}, \quad R \geq R_0$$

which goes to infinite as $R \rightarrow \infty$, so that $\sigma_{ess}(\Delta) = \emptyset$. Moreover,

$$\lambda_0^{-1}(8/r) \leq \inf \left\{ R+1 : R \geq R_0, \frac{c_0^2 k(R)}{4} \geq \frac{8}{r} \right\} = 1 + R_0 \vee k^{-1}(32/c_0^2 r), \quad r > 0.$$

Then by Theorem 2.1 with $\beta_0(r) = c(1 + r^{-d/2})$, we obtain the desired $\beta(r)$ for some $\theta > 0$.

c) If (1.5) holds then by (1), (1.1) holds for

$$\beta(r) = \exp[\theta(1 + r^{-\varepsilon})]$$

for some constant $\theta > 0$. If $\varepsilon \in (0, 1)$ then (1.6) follows from [21, Corollary 3.4(1)]. If $\varepsilon = 1$ then

$$\mu(f^2) \leq r\mu(|\nabla f|^2) + \exp[\theta(1 + r^{-1})]\mu(\varphi_0|f|)^2, \quad r > 0, f \in C_0^1(M).$$

Applying this to $f\varphi_0$ and noting that

$$\frac{1}{2}\mu(\langle \nabla f^2, \nabla \varphi_0^2 \rangle) = -\frac{1}{2}\mu(f^2 L \varphi_0^2) = \lambda_0 \mu_{\varphi_0}(f^2) - \mu(f^2 |\nabla \varphi_0|^2),$$

we arrive at

$$\begin{aligned} \mu_{\varphi_0}(f^2) &\leq r\mu_{\varphi_0}(|\nabla f|^2) + r\mu(f^2 |\nabla \varphi_0|^2) + \frac{r}{2}\mu(\langle \nabla f^2, \nabla \varphi_0^2 \rangle) + e^{\theta(1+r^{-1})}\mu_{\varphi_0}(|f|)^2 \\ &= r\mu_{\varphi_0}(|\nabla f|^2) + r\lambda_0 \mu_{\varphi_0}(f^2) + e^{\theta(1+r^{-1})}\mu_{\varphi_0}(|f|)^2, \quad r > 0. \end{aligned}$$

This implies

$$\mu_{\varphi_0}(f^2) \leq 2r\mu_{\varphi_0}(|\nabla f|^2) + 2e^{\theta(1+r^{-1})}\mu_{\varphi_0}(|f|)^2, \quad r \in (0, 1/(2\lambda_0)).$$

Hence, there exists a constant $\theta' > 0$ such that

$$(3.2) \quad \mu_{\varphi_0}(f^2) \leq r\mu_{\varphi_0}(|\nabla f|^2) + e^{\theta'(1+r^{-1})}\mu_{\varphi_0}(|f|)^2, \quad r > 0.$$

By [20, Corollary 1.1(1)], this is equivalent to the defective log-Sobolev inequality

$$(3.3) \quad \mu_{\varphi_0}(f^2 \log f^2) \leq C_1 \mu_{\varphi_0}(|\nabla f|^2) + C_2, \quad f \in C_b^1(M), \mu_{\varphi_0}(f^2) = 1$$

for some $C_1, C_2 > 0$. On the other hand, (3.2) and the weak Poincaré inequality due to [17, Theorem 3.1] imply the Poincaré inequality (see [17, Proposition 3.1])

$$\mu_{\varphi_0}(f^2) \leq C \mu_{\varphi_0}(|\nabla f|^2) + \mu_{\varphi_0}(f)^2, \quad f \in C_b^1(M)$$

for some constant $C > 0$. Combining this and (3.3) we obtain the strict log-Sobolev inequality, namely, (3.3) with $C_2 = 0$ and some possibly different $C_1 > 0$. Therefore, due to [10], $P_t^{\varphi_0}$ is hypercontractive since it is associated to the Dirichlet form $\mu_{\varphi_0}(\langle \nabla \cdot, \nabla \cdot \rangle)$ on $H^{2,1}(\mu_{\varphi_0})$. We remark that the implication of the hypercontractivity from the defective log-Sobolev inequality can also be deduced by using the uniformly positivity improving property of the diffusion semigroup, see e.g. [1] for details. Then the proof is finished. \square

To prove Theorem 1.2, we first establish the super Poincaré inequality (2.1) for a concrete β_0 .

Lemma 3.1. *In the situation of Theorem 1.2. (1.10) implies (2.1) with $\beta_0(r) = c(1 + r^{-(m+d+1)/2})$ for some constant $c > 0$.*

Proof. By [15, Theorem 5.2] with $\alpha = (m+d+1)/(m+d)$, for any measurable function $f \geq 0$ with $\mu(f) = 1$, we have

$$P_t f(x) \leq (P_{t+s} f(y)) \left(1 + \frac{s}{t}\right)^{(m+d+1)/2} \exp \left[CK(2[\rho_o(x) \vee \rho_o(y)])s + \frac{\alpha \rho(x, y)^2}{4s} \right]$$

for some constant $C > 0$ and all $s, t > 0$. This implies

$$1 = \int_M P_{t+s} f(y) \mu(dy) \geq (P_t f(x)) \left(1 + \frac{s}{t}\right)^{-(m+d+1)/2} \int_{B(x,1)} e^{-CK(2[\rho_o(x) \vee \rho_o(y)])s - \alpha/[4s]} \mu(dy).$$

Taking $s = 1/\sqrt{K(2 + 2\rho_o(x))}$, we obtain

$$P_t f(x) \leq c_0 (1 + t^{-1})^{(m+d+1)/2} \cdot \frac{\exp[c_0 \sqrt{K(2 + 2\rho_o(x))}]}{\mu(B(x, 1))}$$

for some constant $c_0 > 0$ and all $t > 0, x \in M$. Combining this with (1.10) we obtain

$$\|P_t\|_{L^1(\mu) \rightarrow L^\infty(\mu)} \leq c_1 (1 + t^{-1})^{(m+d+1)/2}, \quad t > 0$$

for some constant $c_1 > 0$. According to [20, Theorem 4.5(b)], this is equivalent to (2.1) with $\beta_0(r) = c(1 + r^{-(m+d+1)/2})$ for some constant $c > 0$. \square

Proof of Theorem 1.2. By (1.9) and Cheeger's inequality explained in b) of the proof of Theorem 1.1, we have

$$\lambda_0(R) \geq \frac{\gamma(R)}{4}, \quad R \gg 1.$$

Since $\gamma(R) \rightarrow \infty$ as $R \rightarrow \infty$, the essential spectrum of L is empty and the desired β follows from Theorem 2.1 and Lemma 3.1. The remainder of the proof is then similar to that of Theorem 1.1. \square

4 Proofs of Examples 1.1 and 1.2

Proof of Example 1.1. Since $\text{Sec} \leq -c_2 \rho_o^\delta$ for some $c_2, \delta > 0$ and large ρ_o , Theorem 1.1 implies $\sigma_{\text{ess}}(\Delta) = \emptyset$. Moreover, one may take $K(r) = (d-1)c_1 r^\delta$ and $k(r) = c_2 r^\delta$ for large r , so that

$$k^{-1}(R) \sqrt{K(4 + 2k^{-1}(R))} \leq cR^{\frac{1}{2} + \frac{1}{\delta}}$$

for some constant $c > 0$ and large R . Then the sufficiency and the desired upper bound of $\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})}$ follow from Theorem 1.1.

Next, by the concrete K and Lemma 2.2 we have

$$(4.1) \quad \varphi_0 \geq \frac{1}{C} \exp \left[-C \rho_o^{1+\delta/2} \right]$$

for some constant $C > 0$. If P_t is intrinsically ultracontractive, i.e. $P_t^{\varphi_0}$ is ultracontractive by definition, then, according to [7, Theorem 2.2.4] (see also [10] and [8]), there exists a function $\beta : (0, \infty) \rightarrow (0, \infty)$ such that

$$\mu_{\varphi_0}(f^2 \log f^2) \leq r \mu_{\varphi_0}(|\nabla f|^2) + \beta(r), \quad f \in C_b^1(M), \mu_{\varphi_0}(f^2) = 1.$$

By the concentration of reference measures induced by log-Sobolev inequalities (see e.g. [18, Corollary 6.3]), the above log-Sobolev inequality implies $\mu_{\varphi_0}(e^{\lambda \rho_o^2}) < \infty$ for any $\lambda > 0$. Combining this with (4.1) and noting that the Riemannian volume of a Cartan-Hadamard manifold is infinite, we conclude that $\delta > 2$. Similarly, if P_t is intrinsically hypercontractive, then $\mu_{\varphi_0}(e^{\lambda \rho_o^2}) < \infty$ for some $\lambda > 0$, so that $\delta \geq 2$.

Finally, let $\delta > 2$. If there exists $\theta_1 > 0$ and a positive function h with $h(t) \downarrow 0$ as $t \downarrow 0$ such that

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})} \leq \theta_1 \exp \left[\theta_2 t^{-(\delta+2)/(\delta-2)} \right], \quad t > 0,$$

then [20, Theorem 4.5] implies (1.1) for

$$\begin{aligned}
\beta(r) &= \inf_{s \leq r, t > 0} \frac{s}{t} \|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \rightarrow L^\infty(\mu_{\varphi_0})} e^{t/s-1} \\
&= \theta_1 \inf_{s \leq r, t > 0} \frac{s}{t} \exp \left[h(t) t^{-(\delta+2)/(\delta-2)} + \frac{t}{s} - 1 \right], \quad r > 0.
\end{aligned}$$

Taking $s = r \wedge 1$ and $t = (r^{(\delta-2)/(2\delta)} \wedge 1) h(r^{(\delta-2)/(2\delta)} \wedge 1)^{(\delta-2)/(2\delta)}$, we obtain

$$(4.2) \quad \beta(r) \leq \theta_2 \exp [\tilde{h}(r) r^{-(\delta+2)/(2\delta)}], \quad r > 0$$

for some constant $\theta_2 > 0$ and positive function \tilde{h} with $\tilde{h}(r) \downarrow 0$ as $r \downarrow 0$.

Finally, we aim to deduce from (4.2) that

$$(4.3) \quad \mu(e^{\lambda \rho_o^{1+\delta/2}}) < \infty, \quad \lambda > 0,$$

which is contradictive to (4.1). To this end, we apply [19, Theorem 6.2], which says that

$$(4.4) \quad \mu(\exp[c_1 \rho_o \xi(c_2 \rho_o)]) < \infty$$

holds for some constants $c_1, c_2 > 0$ and

$$\xi(\lambda) := \inf \left\{ s \geq 1 : \int_1^s \frac{1}{t^2} \log \beta(1/(2t^2)) dt \geq \lambda \right\}, \quad \lambda > 0.$$

Since (4.2) implies

$$\begin{aligned}
\int_1^s \frac{1}{t^2} \log \beta(1/(2t^2)) dt &\leq \theta_3 + \theta_3 \int_1^s t^{-(\delta-2)/\delta} \tilde{h}(1/(2t^{(\delta-2)/\delta})) dt \\
&\leq \theta_3 + \varepsilon(s) s^{2/\delta}, \quad s > 1
\end{aligned}$$

for some constant $\theta_3 > 0$ and some positive function ε with $\varepsilon(s) \downarrow 0$ as $s \uparrow \infty$, one has $\xi(\lambda) \lambda^{-\delta/2} \rightarrow \infty$ as $\lambda \rightarrow \infty$. Therefore, (4.3) follows from (4.4). \square

Proof of Example 1.2. Since M is a Cartan-Hadamard manifold and $\delta > 1$, we have

$$L\rho_o \geq \delta \rho_o^{\delta-1} =: \sqrt{\gamma \circ \rho_o}, \quad \rho_o \gg 1.$$

In particular, $\gamma(\infty) = \infty$ so that $\sigma_{ess}(L) = \emptyset$. Moreover, since

$$\text{Ric} \geq -c(1 + \rho_o^{2(\delta-1)}), \quad |\nabla V|^2 = \theta^2 \delta^2 \rho_o^{2(\delta-1)}$$

and $\text{Hess}_V = \theta \text{Hess}_{\rho_o^\delta} \geq 0$ for large ρ_o as M is Cartan-Hadamard, we may take $K(r) = c_1(1 + r^{2(\delta-1)})$ for some constant $c_1 > 0$. Therefore, (1.11) holds for some $c > 0$ and $\varepsilon = \frac{1}{2} + \frac{1}{2(\delta-1)}$. Therefore, the sufficiency follows from Theorem 1.2 provided (1.10) holds. Indeed, since M is a Cartan-Hadamard manifold, we have

$$\mu(B(x, 1)) \geq c(d) \exp \left[\inf_{B(x, 1)} V \right] \geq c(d) \exp[\theta(\rho_o(x) - 1)^\delta], \quad \rho_o(x) \geq 1,$$

where $c(d)$ is the volume of the unit ball in \mathbb{R}^d . This implies (1.10).

On the other hand, by Lemma 2.2 and the concrete K , we have

$$\varphi_0 \geq \frac{1}{C} \exp [-C\rho_o^\delta]$$

for some constant $C > 0$. Then the remainder of the proof is as same as that in the proof of Example 1.1. \square

Acknowledgement. The author would like to thank the referees for useful comments.

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